

# ON THE INTEGERS OF THE FORM $p^2 + b^2 + 2^n$ AND $b_1^2 + b_2^2 + 2^{n^2}$

HAO PAN AND WEI ZHANG

ABSTRACT. We prove that the sumset

$$\{p^2 + b^2 + 2^n : p \text{ is prime and } b, n \in \mathbb{N}\}$$

has a positive lower density. We also construct a residue class with odd modulo, which contains no integer of the form  $p^2 + b^2 + 2^n$ . And similar results are established for the sumset

$$\{b_1^2 + b_2^2 + 2^{n^2} : b_1, b_2, n \in \mathbb{N}\}.$$

## 1. INTRODUCTION

Let  $\mathcal{P}$  denote the set of all primes. In 1934, Romanoff [24] proved that the sumset

$$\mathcal{S}_1 = \{p + 2^n : p \in \mathcal{P}, n \in \mathbb{N}\}$$

has a positive lower density. Subsequently van der Corput [13] proved the complement of  $\mathcal{S}_1$ , i.e.  $\mathbb{N} \setminus \mathcal{S}_1$ , also has a positive lower density. In fact, Erdős [14] showed that every positive integer  $n$  with  $n \equiv 7629217 \pmod{11184810}$  is not of the form  $p + 2^n$ . The key ingredient of Erdős' proof is to find a finite class of residue classes with distinct moduli, which covers all integers. Nowadays, Erdős' idea has been greatly extended, and for the further related developments, the readers may refer to [11, 10, 3, 25, 4, 5, 26, 27, 6, 29, 7, 20, 8, 9, 28].

In 1999, with help of Brüdern and Fouvry's estimations on sums of squares [2], Liu, Liu and Zhan [21] proved a Romanoff-type result:

*The sumset*

$$\mathcal{S}_2 = \{p_1^2 + p_2^2 + 2^{n_1} + 2^{n_2} : p_1, p_2 \in \mathcal{P}, n_1, n_2 \in \mathbb{N}\}$$

*has a positive lower density.*

The key of their proof is the following lemma:

*For  $1 \leq m \leq N$ ,*

$$|\{p_1^2 + p_2^2 - p_3^2 - p_4^2 = m : p_i \in \mathcal{P}, p_i^2 \leq N\}| \ll \mathfrak{S}_-(m) \frac{N}{(\log N)^4},$$

*where  $\mathfrak{S}_-$  will introduced in Section 2.*

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2000 *Mathematics Subject Classification.* Primary 11P32; Secondary 11A07, 11B05, 11B25, 11N36.  
*Key words and phrases.* positive lower density, arithmetical progression, prime, square, power of 2.

In the other direction, recently Crocker [12] proved that there exist infinitely many positive integers not representable as the sum of two squares and two (or fewer) powers of 2.

Motivated by all these results, in the present paper, we shall study the sumset

$$\mathcal{S}_3 = \{p^2 + b^2 + 2^n : p \in \mathcal{P}, n, n_2 \in \mathbb{N}\}$$

First, we have the following Romanoff-type result.

**Theorem 1.1.** *The set  $\mathcal{S}_3$  has a positive lower density.*

Next, we need to say something about the complement of  $\mathcal{S}_3$ . It is not difficult to see that almost all integers in  $\mathcal{S}_3$  are of the form  $4k + 1$  or  $8k + 2$ . However, we shall prove that

**Theorem 1.2.** *There exists a residue class with odd modulo, which contains no integer of the form  $p^2 + b^2 + 2^n$ .*

Since the modulo in Theorem 1.2 is odd, by the Chinese remainder theorem, clearly both the two sets

$$\{x \in \mathbb{N} : x \equiv 1 \pmod{4}, x \notin \mathcal{S}_3\}$$

and

$$\{x \in \mathbb{N} : x \equiv 2 \pmod{8}, x \notin \mathcal{S}_3\}$$

have positive lower densities.

Furthermore, we also have a similar result on the integers not of the form  $p^2 + b^2 - 2^n$ .

**Theorem 1.3.** *There exists a residue class with odd modulo, which contains no integer of the form  $p^2 + b^2 - 2^n$ .*

A well-known result due to Landau [19] asserts that

$$\{b_1^2 + b_2^2 \leq N : b_1, b_2 \in \mathbb{N}\} = \frac{KN}{\sqrt{\log N}}(1 + o(1))$$

where

$$K = \frac{1}{\sqrt{2}} \prod_{\substack{p \in \mathcal{P} \\ p \equiv 3 \pmod{4}}} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}} = 0.764223653 \dots$$

And obviously

$$\{n \in \mathbb{N} : 2^{n^2} \leq N\} \ll \sqrt{\log N}.$$

These facts suggests us to obtain the following results.

**Theorem 1.4.** *The sumset*

$$\mathcal{S}_4 = \{b_1^2 + b_2^2 + 2^{n^2} : b_1, b_2, n \in \mathbb{N}\}$$

*has a positive lower density. And conversely there also exists a residue class with odd modulo, which contains no integer of the form  $b_1^2 + b_2^2 + 2^{n^2}$ .*

The proofs of Theorem 1.1 and the first assertion of Theorem 1.4 are applications of sieve method. And we shall construct a suitable cover of  $\mathbb{Z}$  with odd moduli to prove Theorem 1.2, Theorem 1.3 and the second assertion of Theorem 1.4. Throughout our proof, the implied constants by  $\ll$ ,  $\gg$  and  $O(\cdot)$  will be always absolute.

## 2. PROOF OF THEOREM 1.1

For  $\mathbf{d} = (d_1, d_2, d_3, d_4)$  with  $\mu(\mathbf{d}) := \mu(d_1)\mu(d_2)\mu(d_3)\mu(d_4) \neq 0$ , define

$$A(m, q, \mathbf{d}) = q^{-4} \sum_{\substack{1 \leq a \leq q \\ (a, q) = 1}} e(-am/q) S(q, ad_1^2) S(q, ad_4^2) S(q, -ad_2^2) S(q, -ad_3^2)$$

and

$$\mathfrak{S}(m, \mathbf{d}) = \sum_{q=1}^{\infty} A(m, q, \mathbf{d}),$$

where

$$S(q, a) = \sum_{x=1}^q e(ax^2/q)$$

and  $e(\alpha) = \exp(2\pi\sqrt{-1}\alpha)$ . In particular, we set  $\mathfrak{S}_-(m) = \mathfrak{S}(m, (1, 1, 1, 1))$  and  $\omega(\mathbf{d}, m) = \mathfrak{S}(m, \mathbf{d})/\mathfrak{S}_-(m)$ . By the arguments in [21, Eq. (8.7)], we know

$$\omega(\mathbf{d}, n) = \prod_{\substack{p^u \parallel d_1 d_4 \\ p^v \parallel d_2 d_3}} \omega_{u,v}(p),$$

where  $p^\beta \parallel m$  means  $p^\beta \mid m$  but  $p^{\beta+1} \nmid m$ .

**Lemma 2.1** (Liu, Liu and Zhan [21, Lemma 8.1]). *Suppose that  $p \geq 3$ ,  $p^u \parallel d_1 d_2$  and  $p^v \parallel d_2 d_3$ . If  $p \nmid m$ , then*

$$\omega_{1,0}(p) = \begin{cases} p/(p-1), & \text{if } \left(\frac{-m}{p}\right) = 1, \\ p/(p+1), & \text{if } \left(\frac{-m}{p}\right) = -1, \end{cases}$$

$$\omega_{0,1}(p) = \begin{cases} p/(p-1), & \text{if } \left(\frac{m}{p}\right) = 1, \\ p/(p+1), & \text{if } \left(\frac{m}{p}\right) = -1, \end{cases}$$

and  $\omega_{1,1}(p) = p/(p+1)$ . And if  $p^\beta \parallel m$  for some  $\beta \geq 1$ , then

$$\omega_{1,0}(p) = \omega_{0,1}(p) = \frac{1 + p^{-1} - p^{-\beta} - p^{-\beta-1}}{1 + p^{-1} - p^{-\beta-1} - p^{-\beta-2}}$$

and

$$\omega_{1,1}(p) = \frac{3 - p^{-1} - p^{1-\beta} - p^{-\beta}}{1 + p^{-1} - p^{-\beta-1} - p^{-\beta-2}}.$$

Let

$$\mathcal{A} = \{(x_1, x_2, x_3, x_4) : x_1^2 + x_4^2 = x_2^2 + x_3^2 + m, 1 \leq x_i^2 \leq N\},$$

and for  $\mathbf{d} = (d_1, d_2, d_3, d_4)$ , let

$$\mathcal{A}_{\mathbf{d}} = \{(x_1, x_2, x_3, x_4) \in \mathcal{A} : x_i \equiv 0 \pmod{d_i}\}.$$

**Lemma 2.2** (Brüderer and Fouvry [2, Theorem 3], Liu, Liu and Zhan [21, Lemma 9.1]).

$$|\mathcal{A}_{\mathbf{d}}| = \frac{\omega(\mathbf{d}, m)}{d_1 d_2 d_3 d_4} \frac{\pi}{16} \mathfrak{S}_-(m) \mathfrak{I}(m/N) N + R(m, N, \mathbf{d}),$$

where

$$\mathfrak{I}(\theta) = 2 \int_{\max\{0, -\theta\}}^{\min\{1, 1-\theta\}} t^{-1/2} (1 - \theta - t)^{1/2} dt$$

and

$$\sum_{\substack{d_1, d_2, d_3, d_4 \leq D \\ \mathbf{d} = (d_1, d_2, d_3, d_4) \\ |\mu(d)|=1}} |R(m, N, \mathbf{d})| \ll N^{1-\epsilon}.$$

Let  $D = N^{1/30}$  and  $z = N^{1/300}$ . Define

$$P(z) = \prod_{\substack{p < z \\ p \text{ prime}}} p.$$

Let

$$f(k) = |\{(x_1, x_2, x_3, x_4) \in \mathcal{A} : x_1 x_2 = k\}|.$$

**Lemma 2.3.** For any  $d \mid P(z)$  with  $d \leq \sqrt{D}$ ,

$$\begin{aligned} \sum_{k \equiv 0 \pmod{d}} f(k) &= \frac{\pi}{16} \mathfrak{S}_-(m) \mathfrak{I}(m/N) N \prod_{p \mid d} \left( \frac{\omega_{1,0}(p)}{p} + \frac{\omega_{0,1}(p)}{p} - \frac{\omega_{1,1}(p)}{p^2} \right) \\ &\quad + O\left( \sum_{\substack{d_1, d_2 \mid d, d \mid d_1 d_2 \\ t_1 \mid d/d_1, t_2 \mid d/d_2 \\ \mathbf{d} = (d_1 t_1, d_2 t_2, 1, 1)}} |R(m, N, \mathbf{d})| \right). \end{aligned}$$

*Proof.* Applying Lemma 2.2, we have

$$\begin{aligned}
\sum_{d|k} f(k) &= \sum_{\substack{d_1, d_2 | d \\ d | d_1 d_2}} |\{(x_1, x_2, x_3, x_4) \in \mathcal{A} : (x_1, d) = d_1, (x_2, d) = d_2\}| \\
&= \sum_{\substack{d_1, d_2 | d \\ d | d_1 d_2}} \sum_{\substack{(x_1, x_2, x_3, x_4) \in \mathcal{A} \\ d_1 | x_1, d_2 | x_2}} \left( \sum_{t_1 | (x_1, d)/d_1} \mu(t_1) \right) \left( \sum_{t_2 | (x_2, d)/d_2} \mu(t_2) \right) \\
&= \sum_{\substack{d_1, d_2 | d \\ d | d_1 d_2}} \sum_{\substack{t_1 | d/d_1 \\ t_2 | d/d_2}} \mu(t_1) \mu(t_2) \sum_{\substack{(x_1, x_2, x_3, x_4) \in \mathcal{A} \\ d_1 t_1 | x_1, d_2 t_2 | x_2}} 1 \\
&= \sum_{\substack{d_1, d_2 | d, d | d_1 d_2 \\ t_1 | d/d_1, t_2 | d/d_2 \\ \mathbf{d} = (d_1 t_1, d_2 t_2, 1, 1)}} \mu(t_1) \mu(t_2) \left( \frac{\omega(\mathbf{d}, m)}{d_1 t_1 d_2 t_2} \frac{\pi}{16} \mathfrak{S}_-(m) \mathfrak{I}(m/N) N + R(m, N, \mathbf{d}) \right).
\end{aligned}$$

In view of Lemma 2.1,

$$\begin{aligned}
&\sum_{\substack{d_1, d_2 | d, d | d_1 d_2 \\ t_1 | d/d_1, t_2 | d/d_2 \\ \mathbf{d} = (d_1 t_1, d_2 t_2, 1, 1)}} \mu(t_1) \mu(t_2) \frac{\omega(\mathbf{d}, m)}{d_1 t_1 d_2 t_2} \\
&= \sum_{\substack{[d_1, d_2] = d \\ t_1 | d_2/(d_1, d_2) \\ t_2 | d_1/(d_1, d_2)}} \mu(t_1) \mu(t_2) \prod_{p | t_1 t_2 (d_1, d_2)} \frac{\omega_{1,1}(p)}{p^2} \prod_{p | d_1/(d_1, d_2 t_2)} \frac{\omega_{1,0}(p)}{p} \prod_{p | d_2/(d_1 t_1, d_2)} \frac{\omega_{0,1}(p)}{p} \\
&= \sum_{t_1 t_2 t_3 t_4 t_5 = d} \mu(t_1) \mu(t_2) \prod_{p | t_1 t_2 t_3} \frac{\omega_{1,1}(p)}{p^2} \prod_{p | t_4} \frac{\omega_{1,0}(p)}{p} \prod_{p | t_5} \frac{\omega_{0,1}(p)}{p} \\
&= \prod_{p | d} \left( \frac{\omega_{1,0}(p)}{p} + \frac{\omega_{0,1}(p)}{p} - \frac{\omega_{1,1}(p)}{p^2} \right).
\end{aligned}$$

□

Clearly

$$\begin{aligned}
&|\{(x_1, x_2, x_3, x_4) : x_1^2 + x_4^2 = x_2^2 + x_3^2 + m, 1 \leq x_i^2 \leq N, (x_1 x_2, P(z)) = 1\}| \\
&= \sum_{(k, P(z))=1} f(k) \leq \sum_k f(k) \left( \sum_{d | (k, P(z))} \lambda_d \right)^2 = \sum_{d_1, d_2 | P(z)} \lambda_{d_1} \lambda_{d_2} \sum_{k \equiv 0 \pmod{[d_1, d_2]}} f(k),
\end{aligned}$$

where  $\lambda_d$  are the weights appearing in Selberg's sieve method with  $\lambda_d = 0$  for  $d \geq z$  (cf. [18, Chapter 3]). In view of Lemma 2.3,

$$\begin{aligned} & \sum_{d_1, d_2 | P(z)} \lambda_{d_1} \lambda_{d_2} \sum_{k \equiv 0 \pmod{[d_1, d_2]}} f(k) \\ &= \frac{\pi}{16} \mathfrak{S}_-(m) \mathfrak{I}(m/N) N \sum_{d_1, d_2 | P(z)} \lambda_{d_1} \lambda_{d_2} \prod_{p | [d_1, d_2]} \left( \frac{\omega_{1,0}(p)}{p} + \frac{\omega_{0,1}(p)}{p} - \frac{\omega_{1,1}(p)}{p^2} \right) \\ &+ \sum_{d_1, d_2 | P(z)} \lambda_{d_1} \lambda_{d_2} \cdot O \left( \sum_{\substack{d'_1, d'_2 | [d_1, d_2], [d_1, d_2] | d'_1 d'_2 \\ t_1 | [d_1, d_2] / d'_1, t_2 | [d_1, d_2] / d'_2 \\ \mathbf{d} = (d_1 t_1, d_2 t_2, 1, 1)}} |R(m, N, \mathbf{d})| \right). \end{aligned}$$

By Selberg's sieve method, we have

$$\sum_{d_1, d_2 | P(z)} \lambda_{d_1} \lambda_{d_2} \prod_{p | [d_1, d_2]} \left( \frac{\omega_{1,0}(p)}{p} + \frac{\omega_{0,1}(p)}{p} - \frac{\omega_{1,1}(p)}{p^2} \right) = \frac{1}{G_1(z)},$$

where

$$G_1(z) = \sum_{\substack{d | P(z) \\ d < z}} \prod_{p | d} \frac{\omega_{1,0}(p)p^{-1} + \omega_{0,1}(p)p^{-1} - \omega_{1,1}(p)p^{-2}}{1 - \omega_{1,0}(p)p^{-1} - \omega_{0,1}(p)p^{-1} + \omega_{1,1}(p)p^{-2}}.$$

And since  $|\lambda_d| \leq 1$ ,

$$\begin{aligned} & \sum_{d_1, d_2 | P(z)} \lambda_{d_1} \lambda_{d_2} \cdot O \left( \sum_{\substack{d'_1, d'_2 | [d_1, d_2], [d_1, d_2] | d'_1 d'_2 \\ t_1 | [d_1, d_2] / d'_1, t_2 | [d_1, d_2] / d'_2 \\ \mathbf{d} = (d_1 t_1, d_2 t_2, 1, 1)}} |R(m, N, \mathbf{d})| \right) \\ &\ll \sum_{\substack{d_1, d_2 | P(z) \\ d_1, d_2 < z^2 \\ \mathbf{d} = (d_1, d_2, 1, 1)}} \tau(d_1)^2 \tau(d_2)^2 \tau(d_1 d_2)^2 |R(m, N, \mathbf{d})| \ll N^{1-\epsilon/2}, \end{aligned}$$

where  $\tau$  is the divisor function. Noting that  $\omega_{1,0}(p), \omega_{0,1}(p) = 1 + O(1/p)$  and  $\omega_{1,1}(p) = O(1)$ , we have (cf. [18, Lemma 4.1])

$$G_1(z) \gg \prod_{p < z} \left( 1 + \frac{\omega_{1,0}(p)}{p} + \frac{\omega_{0,1}(p)}{p} - \frac{\omega_{1,1}(p)}{p^2} \right) \gg (\log z)^2.$$

And it had been showed [21, Eq. (2.9)] that

$$\mathfrak{S}_-(m) \ll \prod_{\substack{p^\beta \| m \\ p \geq 3 \\ \beta \geq 0}} \left( 1 + \frac{1}{p} - \frac{1}{p^{\beta+1}} - \frac{1}{p^{\beta+2}} \right).$$

Finally,

$$\begin{aligned}
& |\{(x_1, x_2, x_3, x_4) \in \mathcal{A} : |x_1| < z \text{ or } |x_2| < z\}| \\
& \leq \sum_{\substack{|x_1| < z, x_2^2 \leq N \\ \text{or } |x_2| < z, x_1^2 \leq N}} |\{(x_3, x_4) : x_4^2 - x_3^2 = x_2^2 - x_1^2 + m\}| \\
& \ll \sum_{\substack{|x_1| < z, x_2^2 \leq N \\ \text{or } |x_2| < z, x_1^2 \leq N}} \tau(x_2^2 - x_1^2 + m) \ll N^{2/3} z.
\end{aligned}$$

Thus we obtain that

**Theorem 2.1.** *For a positive integer  $m$ , we have*

$$\begin{aligned}
& |\{(p_1, p_2, b_3, b_4) : p_1^2 + b_4^2 = p_2^2 + b_3^2 + m, p_i \in \mathcal{P}, b_i \in \mathbb{N}, p_i^2, b_i^2 \leq N\}| \\
& \ll \frac{N}{(\log N)^2} \prod_{p|m} \left(1 + \frac{1}{p}\right).
\end{aligned}$$

Now we are ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* By the prime number theorem, clearly

$$|\{p^2 + b^2 \leq N : p \in \mathcal{P}, b \in \mathbb{N}\}| \leq |\mathcal{P} \cap [0, \sqrt{N}]| \cdot |\mathbb{N} \cap [0, \sqrt{N}]| \ll \frac{N}{\log N}.$$

On the other hand, in [23], Rieger proved that

$$|\{p^2 + b^2 \leq N : p \in \mathcal{P}, b \in \mathbb{N}\}| \gg \frac{N}{\log N}.$$

Define

$$r(x) = |\{(p, b, n) : p^2 + b^2 + 2^n = x, p \in \mathcal{P}, b, n \in \mathbb{N}\}|.$$

Recall that  $\mathcal{S}_3 = \{x \in \mathbb{N} : r(x) \geq 1\}$ . Then by the Cauchy-Schwarz inequality,

$$|\{(p, b, n) : p^2 + b^2 + 2^n \leq N, p \in \mathcal{P}, b, n \in \mathbb{N}\}| = \sum_{x \leq N} r(x) \leq \sqrt{|\mathcal{S}_3 \cap [1, N]|} \cdot \sqrt{\sum_{x \leq N} r(x)^2}.$$

Clearly

$$\begin{aligned}
& |\{(p, b, n) : p^2 + b^2 + 2^n \leq N, p \in \mathcal{P}, b, n \in \mathbb{N}\}| \\
& \geq |\{p \in \mathcal{P} : p^2 \leq N/3\}| \cdot |\{b \in \mathbb{N} : b^2 \leq N/3\}| \cdot |\{n \in \mathbb{N} : 2^n \leq N/3\}| \gg N.
\end{aligned}$$

So it suffices to show that

$$\sum_{x \leq N} r(x)^2 \ll N.$$

Applying Theorem 2.1, we have

$$\begin{aligned}
\sum_{x \leq N} r(x)^2 &= |\{(p_1, p_2, b_1, b_2, n_1, n_2) : p_1^2 + b_1^2 + 2^{n_1} = p_2^2 + b_2^2 + 2^{n_2} \leq N\}| \\
&\leq 2 \sum_{2^{n_1} \leq 2^{n_2} \leq N} |\{(p_1, p_2, b_1, b_2) : p_1^2 + b_1^2 - p_2^2 - b_2^2 = 2^{n_2} - 2^{n_1}, p_i^2, b_i^2 \leq N\}| \\
&\ll \frac{N}{\log N} \cdot \frac{\log N}{\log 2} + \sum_{2^{n_1} < 2^{n_2} \leq N} \frac{N}{(\log N)^2} \prod_{p|2^{n_2}-2^{n_1}} \left(1 + \frac{1}{p}\right).
\end{aligned}$$

By Romanoff's arguments [24] (or see [22, p.203]), we know that

$$\sum_{2^{n_1} < 2^{n_2} \leq N} \prod_{p|2^{n_2}-2^{n_1}-1} \left(1 + \frac{1}{p}\right) \ll (\log N)^2.$$

This concludes the proof of Theorem 1.1 □

### 3. PROOF OF THEOREMS 1.2 AND 1.3

For an integer  $a$  and a positive integer  $n$ , let  $a(n)$  denote the residue class  $\{x \in \mathbb{Z} : x \equiv a \pmod{n}\}$ . For a finite system  $\mathcal{A} = \{a_s(n_s)\}_{s=1}^k$ , we say  $\mathcal{A}$  is a cover of  $\mathbb{Z}$  provided that

$$\bigcup_{s=1}^k a_s(n_s) = \mathbb{Z}.$$

Our aim is to find a cover  $\{a_s(n_s)\}_{s=1}^k$  of  $\mathbb{Z}$  and distinct primes  $p_1, p_2, \dots, p_k$  with  $p_s \equiv 3 \pmod{4}$  and  $2^{n_s} \equiv 1 \pmod{p_s}$ . With help of the book [1], the following lemma can be directly verified.



**Lemma 3.1.** *Let*

$$\begin{aligned} \{(a'_s, n'_s, p_s)\}_{s=1}^{49} = & \{(0, 3, 7), (1, 15, 11), (4, 15, 31), (7, 15, 151), (10, 15, 331), \\ & (13, 105, 43), (28, 105, 71), (43, 105, 127), (58, 105, 211), \\ & (73, 105, 29191), (88, 105, 86171), (103, 315, 870031), \\ & (208, 315, 983431), (313, 315, 1765891), (2, 9, 19), (5, 27, 87211), \\ & (14, 81, 71119), (41, 81, 97685839), (68, 81, 163), (23, 135, 271), \\ & (50, 135, 631), (77, 135, 811), (104, 135, 23311), (131, 135, 348031), \\ & (8, 99, 23), (17, 99, 67), (26, 99, 199), (35, 99, 683), (44, 99, 5347), \\ & (53, 99, 599479), (62, 99, 33057806959), (71, 99, 242099935645987), \\ & (80, 495, 991), (179, 495, 2971), (278, 495, 3191), \\ & (377, 495, 48912491), (476, 495, 2252127523412251), (89, 693, 463), \\ & (188, 693, 5419), (287, 693, 14323), (386, 693, 289511839), \\ & (485, 693, 35532364099), (584, 693, 2868251407519807), \\ & (683, 693, 581283643249112959), (98, 297, 694387), \\ & (197, 297, 14973866897175265228063698945547), (296, 891, 1783), \\ & (593, 891, 1409033313878253109224688819), \\ & (890, 891, 12430037668834128259094186647)\} \end{aligned}$$

Then  $\mathcal{A}' = \{a'_s(n'_s)\}_{s=1}^{49}$  is a cover of  $\mathbb{Z}$ . And for  $1 \leq s \leq 49$ , we have  $p_s \mid 2^{n'_s} - 1$  or  $p_s \mid 2^{n'_s} + 1$ .

*Remark.* In [28], Wu and Sun constructed a cover of  $\mathbb{Z}$  with 173 odd moduli and distinct primitive prime divisors.

For  $1 \leq s \leq 49$ , let  $n_s = 2n'_s$  and let  $a_s$  be an integer such that  $a_s \equiv a'_s \pmod{n'_s}$  and  $a_s \equiv 1 \pmod{2}$ . Let  $a_{50} = 0$ ,  $n_{50} = 2$  and  $p_{50} = 3$ . Then by the Chinese remainder theorem,

$$\mathcal{A} = \{a_s(n_s)\}_{s=1}^{50} \tag{*}$$

is a cover of  $\mathbb{Z}$ , and  $2^{n_s} \equiv 1 \pmod{p_s}$  for  $1 \leq s \leq 50$ . Let

$$M_1 = \prod_{s=1}^{50} p_s$$

and let  $\alpha_1$  be an integer such that

$$\alpha_1 \equiv 2^{a_s} \pmod{p_s}$$

for  $1 \leq s \leq 50$ .

Let  $x$  be an arbitrary positive integer with  $x \equiv \alpha_1 \pmod{M_1}$ . Suppose that  $x \in \mathcal{S}_3$ , i.e.,  $x = p^2 + b^2 + 2^n$  for some  $p \in \mathcal{P}$  and  $b, n \in \mathbb{N}$ . Since  $\mathcal{A}$  is a cover of  $\mathbb{Z}$ , there exists  $1 \leq s \leq 50$  such that  $n \equiv a_s \pmod{n_s}$ . Then

$$p^2 + b^2 = x - 2^n \equiv \alpha_1 - 2^{a_s} \equiv 0 \pmod{p_s}.$$

Noting that  $p_s \equiv 3 \pmod{4}$ ,  $-1$  is a quadratic non-residue modulo  $p_s$ . It follows that

$$p \equiv b \equiv 0 \pmod{p_s}.$$

Since  $p$  is prime, we must have  $p = p_s$ .

Below we require some additional congruences. Arbitrarily choose distinct primes  $q_1, q_2, \dots, q_{50}$  such that  $(q_s, M_1) = 1$  and  $q_s \equiv 7 \pmod{8}$  for  $1 \leq s \leq 50$ . Clearly 2 is a quadratic residue and  $-1$  is a quadratic non-residue modulo  $q_s$ . So  $-2^n$  is a quadratic non-residue modulo  $q_s$  for any  $n \geq 0$ . Let

$$M_2 = \prod_{s=1}^{50} q_s,$$

and let  $\alpha_2$  be an integer such that

$$\alpha_2 \equiv p_s^2 \pmod{q_s}$$

for every  $1 \leq s \leq 50$ .

Let  $M = M_1 M_2$ , and let  $\alpha$  be an integer such that

$$\alpha \equiv \alpha_i \pmod{M_i}$$

for  $i = 1, 2$ . Then we have  $\{x \in \mathbb{N} : x \equiv \alpha \pmod{M}\} \cap \mathcal{S}_3 = \emptyset$ . In fact, assume on the contrary that  $x \equiv \alpha \pmod{M}$  and  $x = p^2 + b^2 + 2^n$  for some  $p \in \mathcal{P}$  and  $b, n \in \mathbb{N}$ . Noting that  $x \equiv \alpha_1 \pmod{M_1}$ , we know  $p = p_s$  for some  $1 \leq s \leq 50$ . But since  $x \equiv \alpha_2 \pmod{M_2}$ ,

$$x - p_s^2 - 2^n \equiv \alpha_2 - p_s^2 - 2^n \equiv -2^n \pmod{q_s}.$$

So  $x - p_s^2 - 2^n$  is a quadratic non-residue modulo  $q_s$ , which leads to an evident contradiction since  $x - p_s^2 - 2^n = b^2$ . This concludes the proof of Theorem 1.2.  $\square$

Now let us turn to the proof of Theorem 1.3. We still use the cover  $\mathcal{A}$  in (\*). Now suppose that  $x \equiv -\alpha_1 \pmod{M_1}$  and there exist  $p \in \mathcal{P}$  and  $b, n \in \mathbb{Z}$  such that  $x = p^2 + b^2 - 2^n$ . Then  $n \equiv a_s \pmod{n_s}$  for some  $1 \leq s \leq 50$ , and

$$x + 2^n \equiv -\alpha_1 + 2^{a_s} \equiv 0 \pmod{p_s}.$$

It follows that  $p = p_s$ . The main difficult is to find the additional congruences.

**Lemma 3.2.** *Let*

$$\begin{aligned} \{(c_s, r_s)\}_{s=1}^{50} = & \{(505, 47 \times 178481), (5519, 601 \times 1801), (366, 2731 \times 8191), \\ & (1303, 73 \times 262657), (5149, 233 \times 2089), (5938, 43691 \times 131071), \\ & (182725, 223 \times 616318177), (12153, 174763 \times 524287), \\ & (148671, 13367 \times 164511353), (490297, 431 \times 2099863), \\ & (115115, 2351 \times 13264529), (2370639, 6361 \times 20394401), \\ & (37, 5 \times 17 \times 257), (5615, 13 \times 37 \times 109), (146, 89 \times 397 \times 2113), \\ & (637, 97 \times 241 \times 673), (6393, 103 \times 2143 \times 11119), \\ & (13847, 53 \times 157 \times 1613), (1799, 29 \times 113 \times 15790321), \\ & (335, 59 \times 1103 \times 3033169), (451, 337 \times 92737 \times 649657), \\ & (1479, 641 \times 65537 \times 6700417), (40655, 137 \times 953 \times 26317), \\ & (13353, 228479 \times 48544121 \times 212885833), \\ & (23775, 439 \times 2298041 \times 9361973132609), (10334, 229 \times 457 \times 525313), \\ & (65971, 2687 \times 202029703 \times 1113491139767), (5893, 41 \times 61681 \times 4278255361), \\ & (1344867, 911 \times 112901153 \times 23140471537), (560826, 277 \times 1013 \times 30269), \\ & (406789, 283 \times 4513 \times 165768537521), \\ & (415099, 191 \times 420778751 \times 30327152671), \\ & (61153, 101 \times 4051 \times 8101), (1261375, 307 \times 2857 \times 6529), \\ & (1324442, 107 \times 69431 \times 28059810762433), \\ & (1663519, 321679 \times 26295457 \times 319020217), \\ & (2094571, 3391 \times 23279 \times 65993), (1032375, 571 \times 32377 \times 1212847), \\ & (6321391, 14951 \times 4036961 \times 2646507710984041), \\ & (19031871, 937 \times 6553 \times 7830118297), (7918330, 2833 \times 37171 \times 179951), \\ & (2286429, 61 \times 1321 \times 4562284561), (2227201, 5581 \times 8681 \times 49477), \\ & (207773684, 131 \times 409891 \times 7623851), (2526613, 281 \times 122921 \times 7416361), \\ & (5596695, 433 \times 577 \times 38737), (25234915, 593 \times 1777 \times 25781083), \\ & (7950774, 251 \times 100801 \times 10567201), (10130779, 313 \times 21841 \times 121369), \\ & (14272093, 1429 \times 3361 \times 14449)\}. \end{aligned}$$

*Then for every  $1 \leq s \leq 50$  and  $n \in \mathbb{N}$ ,  $c_s + 2^n$  is a quadratic non-residue modulo  $r_s$ .*

Lemma 3.2 can be checked via a direct computation. In fact, we only need to consider those  $c_s + 2^n$  modulo  $r_s$  for  $0 \leq n < \text{ord}_2(r_s)$ , where  $\text{ord}_2(r)$  denotes the least positive integer such that  $2^{\text{ord}_2(r)} \equiv 1 \pmod{r}$ .

Notice that  $(r_i, M_1) = 1$  and  $(r_i, r_j) = 1$  for any distinct  $i, j$ . Let

$$M_3 = \prod_{s=1}^{50} r_s$$

and let  $\alpha_3$  be an integer such that

$$\alpha_3 \equiv c_s + p_s^2 \pmod{r_s}.$$

Let  $M' = M_1 M_3$  and let  $\alpha'$  be an integer satisfying

$$\alpha \equiv -\alpha_1 \pmod{M_1} \quad \text{and} \quad \alpha \equiv \alpha_3 \pmod{M_3}.$$

For any  $x \equiv \alpha' \pmod{M'}$ , assume on the contrary that  $x = p_s^2 + b^2 - 2^n$  for some  $1 \leq s \leq 50$  and  $b, n \in \mathbb{N}$ . Then

$$b^2 = x + 2^n - p_s^2 \equiv \alpha_3 + 2^n - p_s^2 \equiv c_s + 2^n \pmod{r_s}.$$

This is impossible since  $c_s + 2^n$  is a quadratic non-residue modulo  $r_s$ . Hence the residue class  $\{x \in \mathbb{N} : x \equiv \alpha' \pmod{M'}\}$  contains no integer of the form  $p^2 + b^2 - 2^n$ .  $\square$

*Remark.* Observe that the moduli appear in Lemma 3.2 are all composite. So we have the following problem.

**Problem.** *Does there exist infinitely many primes  $p$  such that the set*

$$\{1 \leq c \leq p : c + 2^n \text{ is a quadratic non-residue modulo } p \text{ for every } n \in \mathbb{N}\}$$

*is non-empty?*

In fact, we don't know any such prime  $p$ . For example, let  $p = 2^{19} - 1$ , then  $27006 + 2^1, 27006 + 2^2, \dots, 27006 + 2^{18}$  are all quadratic non-residues modulo  $p$ , but  $27006 + 2^{19}$  is a quadratic residue modulo  $p$ .

#### 4. THE INTEGERS OF THE FORM $b_1^2 + b_2^2 + 2^{n^2}$

*Proof of the first assertion of Theorem 1.4.* Let

$$\mathcal{Q} = \{x \in \mathbb{N} : x \text{ has no prime factor of the form } 4k + 3\}.$$

Clearly

$$\mathcal{Q} \subseteq \{b_1^2 + b_2^2 : b_1, b_2 \in \mathbb{N}\}.$$

We only need to prove that the set

$$\{x + 2^{n^2} : x \in \mathcal{Q}, n \in \mathbb{N}\}$$

has a positive lower density. As an application of half dimensional sieve method [15], we know that

$$|\{x \in \mathcal{Q} : x \leq N\}| \gg \frac{N}{\sqrt{\log N}}.$$

And with help of Selberg's sieve method, it is not difficult to see that

$$|\{(x_1, x_2) : x_1 = x_2 + m, x_i \in \mathcal{Q}, x_i \leq N\}| \ll \frac{N}{\log N} \prod_{\substack{p|m \\ p \equiv 3 \pmod{4}}} \left(1 + \frac{1}{p}\right)$$

for every positive integer  $M$ . By the Cauchy-Schwarz inequality,

$$\begin{aligned} & |\{x + 2^{n^2} : x + 2^{n^2} \leq N, x \in \mathcal{Q}, n \in \mathbb{N}\}| \\ & \geq \frac{|\{(x, n) : x + 2^{n^2} \leq N, x \in \mathcal{Q}, n \in \mathbb{N}\}|^2}{|\{(x_1, x_2, n_1, n_2) : x_1 + 2^{n_1^2} = x_2 + 2^{n_2^2} \leq N, x_i \in \mathcal{Q}, n_i \in \mathbb{N}\}|}. \end{aligned}$$

So it suffices to show that

$$|\{(x_1, x_2, n_1, n_2) : x_1 + 2^{n_1^2} = x_2 + 2^{n_2^2} \leq N, x_i \in \mathcal{Q}, n_i \in \mathbb{N}\}| \ll N.$$

Now

$$\begin{aligned} & |\{(x_1, x_2, n_1, n_2) : x_1 + 2^{n_1^2} = x_2 + 2^{n_2^2} \leq N, x_i \in \mathcal{Q}, n_i \in \mathbb{N}\}| \\ & \leq |\{(x_1, n_1) : x_1 + 2^{n_1^2} \leq N, x_1 \in \mathcal{Q}, n_1 \in \mathbb{N}\}| \\ & \quad + 2 \sum_{0 \leq n_1 < n_2 \leq \sqrt{\log N / \log 2}} |\{(x_1, x_2) : x_1 - x_2 = 2^{n_2^2} - 2^{n_1^2}, x_i \in \mathcal{Q} \cap [1, N]\}| \\ & \ll N + \frac{N}{\log N} \sum_{0 \leq n_1 < n_2 \leq \sqrt{\log N / \log 2}} \prod_{\substack{p|2^{n_2^2} - 2^{n_1^2} \\ p \equiv 3 \pmod{4}}} \left(1 + \frac{1}{p}\right). \end{aligned}$$

Obviously

$$\begin{aligned} \sum_{0 \leq n_1 < n_2 \leq \sqrt{\log N / \log 2}} \prod_{\substack{p|2^{n_2^2} - 2^{n_1^2} \\ p \equiv 3 \pmod{4}}} \left(1 + \frac{1}{p}\right) & \leq \sum_{0 \leq n_1 < n_2 \leq \sqrt{\log N / \log 2}} \prod_{p|2^{n_2^2} - 2^{n_1^2}} \left(1 + \frac{1}{p}\right) \\ & \leq \sum_d \frac{1}{d} \sum_{\substack{0 \leq n_1 < n_2 \leq \sqrt{\log N / \log 2} \\ n_2^2 \equiv n_1^2 \pmod{\text{ord}_2(d)}}} 1 \end{aligned}$$

Suppose that  $p$  is prime,  $\beta \geq 1$  and  $1 \leq a \leq p^\beta$ . Then we have

$$|\{1 \leq x \leq p^\beta : x^2 \equiv a \pmod{p^\beta}\}| \leq 2p^{\frac{\nu_p(a)}{2}}$$

since the multiplicative group modulo  $p^\beta$  is cyclic, where  $\nu_p(a)$  denotes the greatest integer such that  $p^{\nu_p(a)} \mid a$ . Thus

$$\sum_{\substack{0 \leq n_1 < n_2 \leq \sqrt{\log N / \log 2} \\ n_2^2 \equiv n_1^2 \pmod{\text{ord}_2(d)}}} 1 \ll \sqrt{\frac{\log N}{\log 2}} \left( \sqrt{\frac{\log N}{\log 2}} \cdot \frac{2^{\omega(\text{ord}_2(d))} \sqrt{\text{ord}_2(d)}}{\text{ord}_2(d)} + 1 \right),$$

where  $\omega(r)$  denotes the number of distinct prime factors of  $r$ . We only need to prove that

$$\sum_d \frac{2^{\omega(\text{ord}_2(d))}}{d \sqrt{\text{ord}_2(d)}}$$

converges. Define

$$E(x) = \sum_{k \leq x} \sum_{\text{ord}_2(d)=k} \frac{1}{d}.$$

Romanoff had show that  $E(x) \ll \log x$  (cf. [22, pp. 200-2001]). So

$$\begin{aligned} \sum_d \frac{2^{\omega(\text{ord}_2(d))}}{d \sqrt{\text{ord}_2(d)}} &= \sum_k \frac{2^{\omega(k)}}{\sqrt{k}} \sum_{\text{ord}_2(d)=k} \frac{1}{d} \\ &\ll \int_1^\infty x^{-\frac{1}{3}} d(E(x)) \\ &= x^{-\frac{1}{3}} E(x) \Big|_1^\infty + \frac{1}{3} \int_1^\infty x^{-\frac{4}{3}} E(x) dx = O(1). \end{aligned}$$

Our proof is complete.  $\square$

Let  $\mathcal{A} = \{a_s(n_s)\}_{s=1}^{50}$  be the cover in (\*), and let  $p_1, \dots, p_{50}$  be the corresponding primes with  $p_s \equiv 3 \pmod{4}$  and  $p_s \mid 2^{n_s} - 1$ . Since every  $p_s$  has at least one quadratic non-residue modulo  $p_s$ , the second assertion of Theorem 1.4 is an immediate consequence of the following stronger result.

**Theorem 4.1.** *Let  $\mathcal{N}$  be a set of non-negative integers. Suppose that for every  $1 \leq s \leq 50$ , there exists  $1 \leq e_s \leq p_s$  such that*

$$|\mathcal{N} \cap \{x \in \mathbb{N} : x \equiv e_s \pmod{p_s}\}| < +\infty.$$

*Then there exists a residue class with odd modulo, which contains no integer of the form  $b_1^2 + b_2^2 + 2^n$  with  $n \in \mathcal{N}$ .*

*Proof.* Let  $n_s^* = \text{ord}_2(p_s)$ . Noting that  $n_s^* \mid n_s$  and  $(n_s, p_s) = 1$ , for every  $1 \leq s \leq 50$ , let  $a_s^*$  be an integer such that

$$a_s^* \equiv a_s \pmod{n_s^*}$$

and

$$a_s^* \equiv e_s \pmod{p_s}.$$

Clearly  $\mathcal{A}^* = \{a_s^*(n_s^*)\}_{s=1}^{50}$  is also a cover of  $\mathbb{Z}$ .

Let

$$\mathcal{H}_s = \mathcal{N} \cap \{x \in \mathbb{N} : x \equiv e_s \pmod{p_s}\} = \{h_{s,1}, h_{s,2}, \dots, h_{s,|\mathcal{H}_s|}\}.$$

for  $1 \leq s \leq 50$ . Choose distinct  $|\mathcal{H}_1| + |\mathcal{H}_2| + \dots + |\mathcal{H}_{50}|$  primes

$$q_{1,1}, \dots, q_{1,|\mathcal{H}_1|}, q_{2,1}, \dots, q_{2,|\mathcal{H}_2|}, \dots, q_{50,1}, \dots, q_{50,|\mathcal{H}_{50}|}$$

satisfying that

$$q_{s,t} \equiv 3 \pmod{4}$$

and

$$q_{s,t} \notin \{p_1, p_2, \dots, p_{50}\}$$

for every  $1 \leq s \leq 50$  and  $1 \leq t \leq |\mathcal{H}_s|$ .

Let

$$M^* = \left( \prod_{1 \leq s \leq 50} p_s \cdot \prod_{\substack{1 \leq s \leq 50 \\ 1 \leq t \leq |\mathcal{H}_s|}} q_{s,t} \right)^2$$

and let  $\alpha^*$  be an integer such that

$$\alpha^* \equiv 2^{a_s^*} \pmod{p_s^2}$$

and

$$\alpha^* \equiv 2^{h_{s,t}} + q_{s,t} \pmod{q_{s,t}^2}$$

for every  $s, t$ .

We claim that for any  $x \equiv \alpha^* \pmod{M^*}$ ,  $x$  is not of the form  $b_1^2 + b_2^2 + 2^n$  with  $n \in \mathcal{N}$ . Assume on the contrary that  $x \equiv \alpha^* \pmod{M^*}$  and  $x = b_1^2 + b_2^2 + 2^n$  with  $n \in \mathcal{N}$ . Since  $\mathcal{A}^* = \{a_s^*(n_s^*)\}_{s=1}^{50}$  is a cover, similarly as the arguments in the proof of Theorems 1.2 and 1.3, we know that

$$b_1 \equiv b_2 \equiv 0 \pmod{p_s}$$

for some  $1 \leq s \leq 50$ . It follows that

$$x - 2^n \equiv 2^{a_s^*} - 2^n \equiv 0 \pmod{p_s^2},$$

that is,  $n \equiv a_s^* \pmod{\text{ord}_2(p_s^2)}$ . It is not difficult to check that

$$2^{n_s^*} = 2^{\text{ord}_2(p_s)} \not\equiv 1 \pmod{p_s^2}.$$

(In fact, the only known primes  $p$  with  $2^{p-1} \equiv 1 \pmod{p^2}$  are 1093 and 3511.) And

$$2^{n_s^* p} = \sum_{k=0}^p \binom{p}{k} (2^{n_s^*} - 1)^k \equiv 1 \pmod{p_s^2}.$$

So we must have  $\text{ord}_2(p_s^2) = n_s^* p$ . Consequently

$$n \equiv a_s^* \equiv e_s \pmod{p_s}.$$

Since  $n \in \mathcal{N}$ , we have  $n \in \mathcal{H}_s$  and there exists  $1 \leq t \leq |\mathcal{H}_s|$  such that  $n = h_{s,t}$ . It follows that

$$b_1^2 + b_2^2 = x - 2^n \equiv \alpha^* - 2^{h_{s,t}} \equiv q_{s,t} \equiv 0 \pmod{q_{s,t}^2}.$$

Recalling that  $q_{s,t} \equiv 3 \pmod{4}$ ,  $q_{s,t} \mid b_1^2 + b_2^2$  implies that

$$b_1 \equiv b_2 \pmod{q_{s,t}}.$$

But it is impossible since

$$b_1^2 + b_2^2 \equiv q_{s,t} \not\equiv 0 \pmod{q_{s,t}^2}.$$

□

**Corollary 4.1.** *There exists a positive integer  $m$  such that the set*

$$\{x \in \mathbb{N} : x \text{ is even and } x \text{ is not of the form } b_1^2 + b_2^2 + 2^{mn}\}$$

*contains an infinite arithmetic progression.*

*Proof.* Let  $m = p_1 p_2 \dots p_{50}$  where  $p_1, p_2, \dots, p_{50}$  are the primes in Lemma 3.1. Thus substituting  $\mathcal{N} = \{x \in \mathbb{N} : x \equiv 0 \pmod{m}\}$  and  $e_s = 1$  in Theorem 4.1, we are done. □

**Problem.** *Does there exists a residue class with odd modulo, which contains no integer of the form  $b_1^2 + b_2^2 + 2^n$  with  $b_1, b_2, n \in \mathbb{N}$ ?*

**Acknowledgment.** We are grateful to Professors Hongze Li and Zhi-Wei Sun for their helpful suggestions.

## REFERENCES

- [1] J. Brillhart, D. H. Lehmer, J. L. Selfridge, B. Tuckerman, and S. S. Wagstaff, Jr., *Factorizations of  $b^n \pm 1$ ,  $b = 2, 3, 5, 6, 7, 10, 11, 12$  up to High Powers*, 3rd ed., Contemporary Mathematics 22, Amer. Math. Soc., Providence, RI, 2002.
- [2] J. Brüdern and E. Fouvry, *Lagrange's four squares theorem with almost prime variables*, J. reine angew Math., **454**(1994), 59-96.
- [3] Y.-G. Chen, *On integers of the form  $2^n \pm p_1^{\alpha_1} \dots p_r^{\alpha_r}$* , Proc. Amer. Math. Soc., **128**(2000), 1613-1616.
- [4] Y.-G. Chen, *On integers of the form  $k2^n + 1$* , Proc. Amer. Math. Soc., **129**(2001), 355-361.
- [5] Y.-G. Chen, *On integers of the form  $k - 2^n$  and  $k2^n + 1$* , J. Number Theory, **89**(2001), 121-125.
- [6] Y.-G. Chen, *On integers of the forms  $k^r + 2^n$  and  $k^r 2^n + 1$* , J. Number Theory, **98**(2003), 310-319.
- [7] Y.-G. Chen, *Five consecutive positive odd numbers, none of which can be expressed as a sum of two prime powers*, Math. Comp., **74**(2005), 1025-1031.
- [8] Y.-G. Chen, *On integers of the forms  $k \pm 2^n$  and  $k2^n \pm 1$* , J. Number Theory, **125**(2007), 14-25.
- [9] Y.-G. Chen, R. Feng and N. Templier, *Fermat numbers and integers of the form  $a^k + a^l + p^\alpha$* , Acta Arith., **135**(2008), 51-61.
- [10] F. Cohen and J. L. Selfridge, *Not every number is the sum or difference of two prime powers*, Math. Comp., **29**(1975), 79-81.
- [11] R. Crocker, *On the sum of a prime and two powers of two*, Pacific J. Math., **36**(1971), 103-107.
- [12] R. Crocker, *On the sum of two squares and two powers of  $k$* , Colloq. Math., **112**(2008), 235-267.
- [13] J. G. van der Corput, *On de Polignacs conjecture*, Simon Stevin, **27**(1950), 99-105.
- [14] P. Erdős, *On integers of the form  $2^k + p$  and some related problems*, Summa Brasil. Math. **2**(1950), 113-123.
- [15] H. Iwaniec, *The half dimensional sieve*, Acta Arith., **29**(1976), 69-95.
- [16] H. Iwaniec, *Rosser's sieve*, Acta Arith., **36** (1980), 171-202.
- [17] H. Iwaniec, *A new form of the error term in the linear sieve*, Acta Arith., **37** (1980), 307-320.



- [18] H. Halberstam and H.-E. Richert, *Sieve methods*, London Mathematical Society Monographs, **4**, Academic Press, London-New York, 1974.
- [19] E. Landau, *Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate*, Arch. Math. Phys., **13**(1908), 305-312.
- [20] F. Luca and P. Stănică, *Fibonacci numbers that are not sums of two prime powers*, Proc. Amer. Math. Soc., **133**(2005), 1887-1890.
- [21] J.-Y. Liu, M.-C. Liu and T. Zhan, *Squares of Primes and Powers of 2*, Monatsh. Math., **128**(1999), 283-313.
- [22] M. B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Grad. Texts Math. **165**, Springer-Verlag, New York, 1996.
- [23] G. J. Rieger, *Über die Summe aus einem Quadrat und einem Primzahlquadrat*, J. reine angew. Math., **251**(1968), 89-100.
- [24] N. Romanoff, *Über einige Sätze der additiven Zahlentheorie*, Math. Ann., **109**(1934), 668-678.
- [25] Z. W. Sun, *On integers not of the form  $\pm p^a \pm q^b$* , Proc. Amer. Math. Soc. **128**(2000), 997-1002.
- [26] Z.-W. Sun and M.-H. Le, *Integers not of the form  $c(2^a + 2^b) + p^\alpha$* , Acta Arith., **99**(2001), 183-190.
- [27] Z.-W. Sun and S.-M. Yang, *A note on integers of the form  $2^n + cp$* , Proc. Edinburgh Math. Soc., **45**(2002), 155-160.
- [28] K.-J. Wu and Z.-W. Sun, *Covers of the integers with odd moduli and their applications to the forms  $x^m - 2^n$  and  $x^2 - F_{3n}/2$* , Math. Comp., to appear.
- [29] P. Z. Yuan, *Integers not of the form  $c(2^a + 2^b) + p^\alpha$* , Acta Arith., **115**(2004), 23-28.

*E-mail address:* haopan79@yahoo.com.cn

*E-mail address:* zhangwei\_07@yahoo.com.cn

DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093, PEOPLE'S REPUBLIC OF CHINA